Corrigendum: Spiking Dynamics of Bidimensional Integrate-and-Fire Neurons

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In the original version of the paper *Spiking Dynamics of Bidimensional Integrate-and-Fire Neurons* (SIAM J. Dynamical Systems 8(4)) [3], we unfortunately omitted a condition for the existence of the horizontal asymptote (plateau) in theorem 3.1[1].

In detail, the sixth bullet point of theorem 3.1. shall read:
If $\lim_{v\to-\infty} F'(v) < -2\sqrt{ab} - a$*,* Φ *has a horizontal asymptote (plateau) when* $w \to \infty$.

The proof proceeds exactly as in the original paper, by looking for a domain $\mathcal{B} = \{v \leq v_0, w \leq w_0 + \alpha(v - v_0)\}$ invariant under the dynamics (see equation on the top of p. 1483). A sufficient condition for having a plateau consists in finding parameters α, w_0 and v_0 such that (see eq. (3.5)):

$$H_\alpha(v) = \alpha(F(v) - w + I) - a(bv - w) \leq 0$$

for any $v \leq v_0$ with $w = w_0 + \alpha(v - v_0)$. In the original paper, we upper-bounded $F(v)$ by its minimal value F_{min} and derived an affine upper-bound for $H_\alpha(v) \leq A(v - v_0) + B$ with $A < 0$, which does not ensure negativity for all $v \leq v_0$.

This can be achieved using the convexity of F and under the additional condition stated. Indeed, strict convexity of F implies that for all $v < v_0$, $F(v) > F(v_0) - \xi(v - v_0)$, with $\xi = -F'(v_0) > 0$ for v_0 strictly smaller than the value associated with the minimum of F. Let us now choose α such that

$$\lim_{v\to-\infty} F'(v) < -\xi = F'(v_0) < \alpha < 0.$$

We obtain:

$$\begin{aligned} H_\alpha(v) \quad &\leq \alpha(F(v_0) - \xi(v - v_0) - w_0 - \alpha(v - v_0) + I) - a(bv - w_0 - \alpha(v - v_0)) \\ &\leq |v - v_0|(\alpha^2 + \alpha(\xi - a) + ab) + H_\alpha(v_0) \end{aligned}$$

(the absolute value, valid only for $v < v_0$, highlights the positivity of the term in v). A sufficient condition is to find parameters such that $H_\alpha(v_0) < 0$ and:

$$\alpha^2 + \alpha(\xi - a) + ab < 0$$

[1] This correction also applies to [1, 2]

Because this polynomial has a positive leading coefficient, finding an $\alpha < 0$ satisfying this condition requires (1) that the polynomial has two roots, and (2) that α falls between those roots, i.e. at least one of the roots is strictly negative. This provides the condition on $\lim_{v\to-\infty} F'(v)$. Indeed, for these conditions to apply, we need that:

1. The discriminant of the quadratic equation shall be strictly positive:

$$(\xi - a)^2 > 4ab,$$

   i.e. either (i) $\xi > a + 2\sqrt{ab}$ or (ii) $0 < \xi < a - 2\sqrt{ab}$.

2. at least one of the roots is strictly negative, i.e.:

$$-(\xi - a) - \sqrt{(\xi - a)^2 - 4ab} < 0$$

   In case (i), since $\xi - a > 0$, the above inequality is readily satisfied. In case (ii), the first term is positive, and we thus need:

$$0 < (a - \xi) < \sqrt{(\xi - a)^2 - 4ab}$$

   which is never satisfied because $ab > 0$.

From this point, we can conclude, exactly as in the original proof, by choosing v_0 such that $F'(v_0) < -a - 2\sqrt{ab}$ and w_0 small enough such that $H_\alpha(v_0) < 0$, ensuring the existence of a plateau for Φ under the condition $\lim_{v\to-\infty} F'(v) < -2\sqrt{ab} - a$.

References

[1] J. E. Rubin, J. Signerska-Rynkowska, J. D. Touboul, and A. Vidal. Wild oscillations in a nonlinear neuron model with resets:(i) bursting, spike adding and chaos. *Discrete & Continuous Dynamical Systems-B*, 22(3967-4002), 2017.

[2] J. E. Rubin, J. Signerska-Rynkowska, J. D. Touboul, and A. Vidal. Wild oscillations in a nonlinear neuron model with resets:() mixed-mode oscillations. *Discrete & Continuous Dynamical Systems-B*, 22(10):4003–4039, 2017.

[3] Jonathan Touboul and Romain Brette. Spiking dynamics of bidimensional integrate-and-fire neurons. *SIAM Journal on Applied Dynamical Systems*, 8(4):1462–1506, 2009.